LETTERS TO THE EDITOR

OSCILLATIONS IN A BINGHAM BODY

The note by Hanks (1973) on the problem of motion generated by a plate contacting a Bingham body presents a solution which is necessarily wrong because the constitutive description in Hank's Equation (1) is not that of a Bingham body, nor of any other rational material. The error is in the sign of the yield stress term. The correct Bingham body constitutive description for the kinematics of this problem is (Bird et al., 1960)

$$\tau_{xy} = -\eta \frac{\partial v_x}{\partial y} - \tau_0 \cdot \text{sign}\left(\frac{\partial v_x}{\partial y}\right) \quad \text{if} \quad |\tau_{xy}| > \tau_0 \tag{1}$$

$$\frac{\partial v_x}{\partial y} = 0 \quad \text{if} \quad |\tau_{xy}| \leq \tau_0$$

For shear to occur, the stress must either be greater than the yield stress τ_0 or less than $-\tau_0$. The resulting equation of motion (2) is then also incorrect in that it only applies when the magnitude of the shear stress exceeds the yield stress. The complete description must account for the fact that with an oscillating plate the stress oscillates through zero and that during part of the cycle the stress is necessarily less than the yield stress:

$$\rho \frac{\partial v_x}{\partial t} = \eta \frac{\partial^2 v_x}{\partial y^2} \quad \text{if} \quad |\tau_{xy}| > \tau_0 \tag{2}$$

$$\rho \frac{\partial v_x}{\partial t} = -\frac{\partial \tau_{xy}}{\partial y}, \quad \frac{\partial v_x}{\partial y} = 0 \quad \text{if} \quad |\tau_{xy}| \leq \tau_0$$

Equations (1) and (2) constitute a very complicated system which is not amenable to the same simple solution techniques used for the analogous Newtonian problem. It is incorrect, moreover, to assume as Hanks does that shear only occurs in a region $0 \leq y \leq h$ near the plate. Should an unsheared rigid region $h < y < \infty$ exist, it could not simply oscillate, since an oscillation of nonzero amplitude throughout all space requires an oscillatory applied force of infinite amplitude. It follows that since velocity and stress are continuous quantities, both v_x and $\frac{\partial v_x}{\partial y}$ should vanish at the hypothetical $y = h$ and since v_x is prescribed at $y = 0$, the system would be overdetermined. Indeed, the only nonparadoxical possibility is that $h \rightarrow \infty$, that is, that no unsheared region exists.

The oscillation of a plate next to any viscous fluid with a yield stress must result in a motion which extends to infinity, but which decreases in amplitude with distance from the plate, as in the case of a Newtonian fluid. This can be better appreciated by examining a time-dependent flow which can be solved exactly—the accelerating plate which corresponds in form to the start-up of an oscillating plate. Suppose that the plate in contact with a quiescent Bingham body imposes a change in stress at $t = 0$ from $\tau_{xy} = 0$ to $\tau_{xy} = -T$ at $y = 0$. If $T < \tau_0$, it is clear that no motion will occur and that $\frac{\partial \tau_{xy}}{\partial y} = 0$ [from Equation (B)] and $\tau_{xy} = -T$ everywhere (the solid is perfectly rigid). If $T > \tau_0$, then the equations describing the motion are

$$\rho \frac{\partial v_x}{\partial t} = -\frac{\partial \tau_{xy}}{\partial y}; \quad \tau_{xy} = -\eta \frac{\partial v_x}{\partial y} - \tau_0$$

or

$$\rho \frac{\partial \tau_{xy}}{\partial t} = \eta \frac{\partial^2 \tau_{xy}}{\partial y^2} \tag{3}$$

with boundary conditions

$$\tau_{xy} = -T \quad \text{at} \quad y = 0$$

$$\tau_{xy} \rightarrow -\tau_0 \quad \text{as} \quad y \rightarrow \infty$$

That the latter is a proper boundary condition is clear when one notes that as the stress at $y = 0$ goes from $\tau_{xy} = 0$ to $\tau_{xy} = -\tau_0$, the stress everywhere in the body is exactly that at $y = 0$. Thus when the plate begins to move ($|\tau_{xy}| > \tau_0$ at $y = 0$), the stress far from the plate is at $\tau_{xy} = -\tau_0$. Solution of Equation (3) with the boundary conditions given above is

$$\tau_{xy} = -\tau_0 - (\mathrm{T} - \tau_0)\,\text{erfc}\left(\frac{y}{\sqrt{4\eta t/\rho}}\right) \tag{4}$$

and

$$v_x = -\sqrt{\frac{4\eta t}{\rho}}\left(\frac{T - \tau_0}{\eta}\right) \int_{\frac{y}{\sqrt{\frac{4\eta t}{\rho}}}}^{\infty} \text{erfc}\,(\xi)\, d\xi$$

Thus, for all time after the initial imposition of the stress, the velocity v_x is nonzero for all y. The only difference from the Newtonian solution is the offset τ_0 *in the stress. It seems* quite natural then that the oscillating plate should result in shear regions of decreasing amplitude extending to infinity for both Bingham and Newtonian materials.

Finally, it should be noted that it nas been assumed that the subyield Bingham material is perfectly rigid. If this is not so, that is, if the elastic modulus of the subyield material is finite (as suggested by the formulation of Oldroyd, 1947), then the velocity of propagation of stress will be finite and equal to the sonic velocity of the solid. For the oscillating plate, stress-displacement- and velocity-amplitudes for an elastic solid are travelling waves, undamped to infinity. The relations are, for a solid whose constitutive relation is $\tau_{xy} = -G \frac{\partial X}{\partial Y}$,

$$\tau_{xy} = T_1 \cos\left[\omega\left(t - \sqrt{\frac{\rho}{G}}\, y\right)\right]$$

$$v_x = \frac{\partial X}{\partial t} = \frac{T_1 \omega}{\sqrt{\rho G}} \cos\left[\omega\left(t - \sqrt{\frac{\rho}{G}}\, y\right)\right] \tag{5}$$

where ω is the frequency and T_1 is the amplitude of stress oscillation; X is the displacement of a material element relative to its rest position, and G is the elastic modulus of the solid. If the subyield Bingham body is elastic, it follows that a velocity oscillation will result in fluid shear only if the amplitude of the oscillation is greater than $\frac{\tau_0 \omega}{\sqrt{\rho G}}$.

LITERATURE CITED

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